Discrete Mathematics 89 (1991) 315–317
North-Holland

Note

A note on vector representation of graphs

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Received 29 November 1988
Revised 13 June 1989

Abstract

Šiňajová, E., A note on vector representation of graphs, Discrete Mathematics 89 (1991) 315–317.

Parsons and Pisanski [1] introduced an orthogonal representation of graphs in the following way. Let G be a simple graph with vertices $x_1, \ldots, x_n$. An orthogonal real (rational) representation of G is a list of nonzero vectors $\bar{x}_1, \ldots, \bar{x}_n$ in $\mathbb{R}^d$ (in $\mathbb{Q}^d$) such that for all i, j $(1 \leqslant i < j \leqslant n)$ the inner product $\bar{x}_i\bar{x}_j$ is negative or zero according as the vertex i is adjacent to or not adjacent to the vertex x_j. The least dimension d necessary for such a representation is denoted by $d(G, \mathbb{R})$ or $d(G, \mathbb{Q})$, respectively.

Note that geometrical dimensions of graphs are defined by various ways using inner product in e.g., Lovász [2], Maehara [3–4], Rödl, Reiterman and Šiňajová [4–5].

Parsons and Pisanski asked the following:

$$\text{Is } d(G, \mathbb{R}) = d(G, \mathbb{Q}) \quad \text{for every graph } G?$$

They observed that $n/2 \leqslant d(G, \mathbb{R}) \leqslant n$ for every graph G on n vertices. In the present note we are going to give an explicit construction of an optimal representation for each graph G and prove that $d(G, \mathbb{R}) = d(G, \mathbb{Q})$.

Theorem. *Let G be a connected graph on n vertices, $n \geqslant 2$. Then*

$$d(G, \mathbb{R}) = d(G, \mathbb{Q}) = n - 1.$$

Proof. As $d(G, \mathbb{R}) \leqslant d(G, \mathbb{Q})$ for each G it is sufficient to prove (1) $d(G, \mathbb{R}) \geqslant n-1$ and (2) $d(G, \mathbb{Q}) \leqslant n-1$. We prove (1) by induction. The case $n=2$ is clear. Let $n \geqslant 3$ and assume G can be represented in $\mathbb{R}^{n-2}$ by vectors $\bar{x}_1, \ldots, \bar{x}_n$. As G is connected, we can find a vertex, say x_n, in such a way that $G - x_n$ is again connected. Put

$$\bar{u} = \frac{\bar{x}_n}{\|\bar{x}_n\|} \quad \text{and} \quad \tilde{x}_i = \bar{x}_i - (\bar{x}_i\bar{u})\bar{u}, \quad i = 1, \ldots, n-1.$$

Then $\tilde{x}_1, \ldots, \tilde{x}_{n-1}$ lie in the hyperplane $\bar{u}\bar{x} = 0$ and thus lie in a subspace of dimension $n-3$.

We shall show that (a) $\tilde{x}_i \neq \bar{0}$ and (b) $\tilde{x}_i\tilde{x}_j \leqslant \bar{x}_i\bar{x}_j \leqslant 0$ for $i \neq j$, $i, j = 1, \ldots, n-1$.

(a) If $\tilde{x}_i = \bar{0}$, then $\bar{0} \neq \bar{x}_i = (\bar{x}_i\bar{u})\bar{u}$ and $\bar{x}_i\bar{u} \neq 0$. Let $j \neq i, n$. Then $\bar{x}_j\bar{x}_i = (\bar{x}_i\bar{u})(\bar{u}\bar{x}_j) \geqslant 0$, and thus $\bar{x}_j\bar{x}_i = 0$. On the other hand

$$\bar{x}_j\bar{x}_n = \bar{x}_j \|\bar{x}_n\| \bar{u} = \|\bar{x}_n\| \frac{1}{\bar{x}_i\bar{u}} \bar{x}_i\bar{x}_j = \frac{\|\bar{x}_n\|}{\bar{x}_i\bar{u}} (\bar{x}_j\bar{x}_i) \geqslant 0$$

and thus $\bar{x}_j\bar{x}_n = 0$. This means that x_ix_n is an isolated edge, a contradiction.

(b) Now

$$\tilde{x}_i\tilde{x}_j = (\bar{x}_i - (\bar{x}_i\bar{u})\bar{u})(\bar{x}_j - (\bar{x}_j\bar{u})\bar{u}) = \bar{x}_j\bar{x}_i - (\bar{x}_i\bar{u})(\bar{x}_j\bar{u}) \leqslant \bar{x}_j\bar{x}_i \leqslant 0.$$

We have shown that $\tilde{x}_1, \ldots, \tilde{x}_{n-1}$ form a representation of some graph G_1 on $n-1$ vertices in subspace of dimension $n-3$. Moreover, if x_ix_j is an edge in G, then x_ix_j is an edge in G_1. Thus G_1 is also connected. This is a contradiction to the induction hypothesis.

(2) Choose vertex x_1 and arrange vertices according to their distances from x_1. We shall construct the rational representation of induced connected subgraphs G_i of graph G on vertices $\{x_1, \ldots, x_i\}$ $(i = 2, \ldots, n)$ in $\mathbb{Q}^{i-1}$ by induction on i. For $i = 2$, put $\bar{x}_1 = 1$ and $\bar{x}_2 = -1$. Let $\bar{x}_1, \ldots, \bar{x}_k$ be any rational representation of G_k in $\mathbb{Q}^{k-1}$. From part (1) of the proof, the space spanned by $\bar{x}_1, \ldots, \bar{x}_k$ has dimension $k-1$. Without loss of generality we may assume that $\bar{x}_k$ is a linear combination of $\bar{x}_1, \ldots, \bar{x}_{k-1}$ and thus $\bar{x}_1, \ldots, \bar{x}_{k-1}$ are linearly independent. To form a representation of G_{k+1} put

$$\tilde{x}_i = (\bar{x}_i, 0), \quad (i \leqslant k-1)$$

$$\tilde{x}_k = (\bar{x}_k, 1) \quad \text{and} \quad \tilde{x}_{k+1} = (\alpha_1, \ldots, \alpha_k),$$

where $(\alpha_1, \ldots, \alpha_k)$ is the solution of the equation

$$A \begin{pmatrix} \alpha_n \\ \vdots \\ \alpha_k \end{pmatrix} = \begin{pmatrix} \varepsilon_n \\ \vdots \\ \varepsilon_k \end{pmatrix}$$

where $\varepsilon_i = -1$ if x_ix_{k+1} is an edge, $\varepsilon_i = 0$ if x_ix_{k+1} is not an edge and A is a k by k matrix, the ith row of which is the vector $\tilde{x}_i$. As A is nonsingular and the right hand side is a nonzero vector and moreover both are rational, a solution

$(\alpha_1, \ldots, \alpha_k)$ does exist and is a rational nonzero vector. It is easy to verify that $\bar{x}_i (i = 1, \ldots, k+1)$ form a rational representation of G_{k+1}. Of course, the representation of G_n is a desired rational representation of G. □

Corollary. *The number of vertices of G minus the number of components with at least two vertices equals* $d(G, \mathbb{R}) = d(G, \mathbb{Q})$.

Proof. Given a representation of a graph, vertices corresponding to vectors of distinct components are orthogonal. Thus, a representation is (and can be constructed as) an orthogonal sum of representations of components. A component on k vertices ($k \geq 2$) has dimension $k-1$, and an isolated vertex has dimension 1. Thus, the dimension of all of G is n minus the number of components with at least two vertices. □

References

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